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An overlooked aspect of the wind-driven oceanic circulation

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The wind-driven circulation of a simple model of the oceanic circulation (linear and homogeneous) is investigated in detail to delineate the role of the Ekman layer mass flux in driving hitherto overlooked components of the oceanic circulation.

The role of upwelling boundary-layer regions in driving interior geostrophic circulations is discussed in detail. Several interesting circulations hidden in the earlier transport theories are described.

1. Introduction

Theories of the wind-driven oceanic circulation (e.g. Sverdrup 1947; Stommel 1948; Munk 1950, etc.) usually relate the horizontal mass transport to the applied wind stress. As Stommel (1955) pointed out, this powerful technique of dealing solely with the vertical integral of the horizontal velocity often hides important information concerning the partition of the flow between the surface viscous boundary layer (the Ekman layer) and the geostrophic, inviscid interior. Certain important questions dealing with the transport of mass in the Ekman layer, which may be as large as the total interior transport, are not dealt with by the transport theories.

It is known, for example, that the transport in the Ekman layer is directed (in the northern hemisphere) to the right and perpendicular to the applied stress. What happens to this flow when it impinges on an ocean boundary? What happens if the stress is such as to produce a horizontally non-divergent Ekman layer? In such a case the flow is limited to the Ekman layer with the fluid below the Ekman layer remaining at rest. Are western boundary currents analogous to the Gulf Stream still required? And if so, are such currents also as shallow as the Ekman layer?

The purpose of this paper is to construct a model sufficiently simple to be able to answer such questions by explicitly dealing with the Ekman layer and the region below the Ekman layer separately. In doing so, certain novel circulation phenomena emerge which are lost by vertical averaging. To keep the model simple enough for this purpose a linear model of a closed, homogeneous ocean is considered. Nevertheless, the general qualitative results are not expected to depend strongly on these simplifying assumptions.

It will be shown that the interior geostrophic circulations depend on the stress itself in distinction to the transports which depend entirely on the curl of the wind stress. It appears, for example, that the interior geostrophic zonal flow need not be zero on the eastern boundary of the oceanic basin although the total transport must be. In fact the simple result is really that the horizontal flux in the Ekman layer, as well as its divergence, can drive a geostrophic interior circulation and intense boundary currents.

Since the vertical flux balance is important in determining the complete oceanic circulation it is necessary to consider in detail the boundary layers in which the vertical mass flux balance is achieved. Aside from these upwelling regions, which girdle the entire basin, the elements of the theory are fairly standard.

2. The model

Consider a rectangular oceanic basin of constant depth D, bounded by rigid horizontal walls on $x = 0, L$, and $y = 0, bL$. The fluid is incompressible and homogeneous with a density ρ. The effect of the earth's sphericity is modelled by using a variable Coriolis parameter, f, equal to twice the local normal component of the earth's rotation. It is taken as a linear function of y, the northward co-ordinate, i.e.

$$f = f_0 + \beta_* y.$$

The equations of motion are

$$\begin{aligned} uu_x + vu_y + wu_z - fv &= \frac{-p_x}{\rho} + \nu_H(u_{xx} + u_{yy}) + \nu_V u_{zz}, \\ uv_x + vv_y + wv_z + fu &= \frac{-p_y}{\rho} + \nu_H(v_{xx} + v_{yy}) + \nu_V v_{zz}, \\ uw_x + vw_y + ww_z &= \frac{-p_z}{\rho} + \nu_H(w_{xx} + w_{yy}) + \nu_V w_{zz}, \\ u_x + v_y + w_z &= 0. \end{aligned}$$

The co-ordinates x, y and z measure eastward, northward and upward respectively while u, v and w are the corresponding velocity components. Constant momentum Austausch coefficients have been introduced, ν_H for horizontal mixing, ν_V for vertical mixing. The general results of this paper do not depend crucially on their relative sizes. The ocean is driven on its surface by a stress $\boldsymbol{\tau}(x, y) = \hat{\imath}\tau^{(x)} + \hat{\jmath}\tau^{(y)}$; $\hat{\imath}$ and $\hat{\jmath}$ being unit vectors parallel to the x- and y-axes respectively.

Dimensionless variables (primed) are introduced as follows

$$\begin{bmatrix} u \\ v \end{bmatrix} = U \begin{bmatrix} u' \\ v' \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = L \begin{bmatrix} x' \\ y' \end{bmatrix},$$

$$w = \frac{D}{L} U w', \quad z = Dz',$$

$$p = \rho U L f_0 p', \quad \boldsymbol{\tau} = \tau_0 \boldsymbol{\tau}',$$

$$f; f_0(1 + \beta y') = f_0 f',$$

where $U = \tau_0/(\nu_V \rho f_0/2)^{\frac{1}{2}}$, $\beta = \beta_* L/f_0$, while τ_0 is a characteristic value for the applied wind stress.

In dimensionless units the equations of motion become, after dropping the prime notation,

$$\epsilon(uu_x + vu_y + wu_z) = -p_x + fv + \frac{E_H}{2}(u_{xx} + u_{yy}) + \frac{E_V}{2}u_{zz}, \tag{2.1a}$$

$$\epsilon(uv_x + vv_y + wv_z) = -p_y - fu + \frac{E_H}{2}(v_{xx} + v_{yy}) + \frac{E_V}{2}v_{zz}, \tag{2.1b}$$

$$\delta^2\epsilon(uw_x + vw_y + ww_z) = -p_z + \frac{\delta^2 E_H}{2}(w_{xx} + w_{yy}) + \frac{\delta^2 E_V}{2}w_{zz}, \tag{2.1c}$$

$$u_x + v_y + w_z = 0. \tag{2.1d}$$

Four dimensionless parameters have been introduced:

$\epsilon = U/f_0 L$, the Rossby number;
$E_V = 2\nu_V/f_0 D^2$, the 'vertical' Ekman number;
$E_H = 2\nu_H/f_0 L^2$, the 'horizontal' Ekman number;
$\delta = D/L$, the aspect ratio.

Only the linear problem will be considered, so that terms multiplied by ϵ in (2.1) will immediately be dropped. Both E_H and E_V are small, and for simplicity of exposition only, we shall arbitrarily take $E_H = E_V \equiv E$. No preference is therefore given to either horizontal or vertical turbulent mixing in the theory. Distinct values for E_H and E_V, uncertain at best, can be carried along without changing the essential nature of the results. Finally, the aspect ratio δ is also considered small.

The boundary conditions are:

$$\begin{aligned} (u, v, w) &= 0 \quad \text{on} \quad x = 0, 1, \\ y &= 0, b, \\ \text{and} \quad z &= 0; \\ w &= 0, \\ \begin{bmatrix} u_z \\ v_z \end{bmatrix} &= E^{-\frac{1}{2}} \begin{bmatrix} \tau^{(x)} \\ \tau^{(y)} \end{bmatrix} \quad \text{on} \quad z = 1. \end{aligned}$$

3. The upper Ekman layer

In regions removed from the effects of the side walls the dynamical fields can be represented as follows;

$$u = E^{\frac{1}{2}}u_I(x, y, z) + \dots \quad + u_E(x, y, \zeta) + \dots, \tag{3.1a}$$

$$v = E^{\frac{1}{2}}v_I(x, y, z) + \dots \quad + v_E(x, y, \zeta) + \dots, \tag{3.1b}$$

$$w = E^{\frac{1}{2}}w_I(x, y, z) + \dots \quad + E^{\frac{1}{2}}w_E(x, y, \zeta) + \dots, \tag{3.1c}$$

$$p = E^{\frac{1}{2}}p_I(x, y, z) + \dots \quad + \delta^2 E p_E(x, y, \zeta) + \dots. \tag{3.1d}$$

The variables are composed of two parts; subscripted I variables represent the fields below the Ekman layer, while the E subscripted variables represent the corrections needed within the Ekman layer region, i.e. a distance of $O(E)^{\frac{1}{2}}$ of

$z = 1$. The boundary-layer corrections are functions of

$$\zeta = (1-z)E^{-\frac{1}{2}},$$

and go to zero as ζ becomes large.

The Ekman layer problem is a standard one in oceanography and yields the following results:

$$u_E = \frac{1}{2\sqrt{f}} e^{-\zeta\sqrt{f}}[(\tau^{(y)} - \tau^{(x)}) \sin \zeta\sqrt{f} + (\tau^{(y)} + \tau^{(x)}) \cos \zeta\sqrt{f}], \tag{3.2a}$$

$$v_E = \frac{1}{2\sqrt{f}} e^{-\zeta\sqrt{f}}[(\tau^{(y)} - \tau^{(x)}) \cos \zeta\sqrt{f} - (\tau^{(y)} + \tau^{(x)}) \sin \zeta\sqrt{f}], \tag{3.2b}$$

$$w_E(x, y, 0) = -\tfrac{1}{2}\hat{\mathbf{k}}.\ \text{curl}\ (\boldsymbol{\tau}/f), \tag{3.2c}$$

where $\hat{\mathbf{k}}$ is a unit vertical vector.

The mass flux vector, $\mathbf{U}_E = \hat{\mathbf{i}}U_E + \hat{\mathbf{j}}V_E$, representing the mass flux within the Ekman layer is

$$\mathbf{U}_E = E^{\frac{1}{2}}\int_0^\infty d\zeta(\hat{\mathbf{i}}u_E + \hat{\mathbf{j}}v_E) = \frac{E^{\frac{1}{2}}}{2f}[\hat{\mathbf{i}}\tau^{(y)} - \hat{\mathbf{j}}\tau^{(x)}], \tag{3.2d}$$

while the application of (2.2*b*) with the use of (3.2*c*) yields as a boundary condition for the interior flow

$$w_I(x, y, 1) = \tfrac{1}{2}\hat{\mathbf{k}}.\text{curl}\ (\boldsymbol{\tau}/f). \tag{3.2e}$$

4. The interior

In regions not adjacent to the lateral boundaries, and below the Ekman layer, the dynamic variables are represented by the I subscripted variables and to $O(E)$ satisfy the equations:

$$0 = -p_{Ix} + fv_I, \tag{4.1a}$$

$$0 = -p_{Iy} - fu_I, \tag{4.1b}$$

$$0 = -p_{Iz}, \tag{4.1c}$$

$$0 = u_{Ix} + v_{Iy} + w_{Iz}. \tag{4.1d}$$

The motion is geostrophic and hydrostatic and therefore the horizontal velocities are independent of depth. By eliminating the pressure we obtain the Sverdrup relation,

$$\beta v_I = fw_{Iz}. \tag{4.2}$$

Since the interior velocity is $O(E^{\frac{1}{2}})$ the suction velocity out of the lower Ekman layer is only $O(E)$ (Pedlosky & Greenspan 1967) so that

$$w_I(x, y, 0) = 0. \tag{4.3}$$

It follows that

$$u_I = \tfrac{1}{2}f\hat{\mathbf{k}}.\text{curl}\ (\boldsymbol{\tau}/f), \tag{4.4a}$$

$$w_I = \tfrac{1}{2}z\hat{\mathbf{k}}.\text{curl}\ (\boldsymbol{\tau}/f). \tag{4.4b}$$

The northward transport in the geostrophic interior,

$$E^{\frac{1}{2}}\int_0^1 v_I dz = V_I,$$

is $E^{\frac{1}{2}}v_I$ (since v_I is independent of z) so that the total northward mass transport is given by the familiar Sverdrup transport relation

$$(V_I + V_E) = \tfrac{1}{2}E^{\frac{1}{2}}\hat{\mathbf{k}}\,.\,\mathrm{curl}\,\boldsymbol{\tau}. \tag{4.5}$$

From (4.1 *a*) and (4.1 *b*) we deduce that

$$p_I = -\frac{1}{2}\frac{f^2}{\beta}\int_x^1 \hat{\mathbf{k}}\,.\,\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right)dx' + h(y), \tag{4.6a}$$

$$u_I = \frac{1}{2f}\frac{\partial}{\partial y}\left[\frac{f^2}{\beta}\int_x^1 \hat{\mathbf{k}}\,.\,\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right)dx'\right] - \frac{1}{f}\frac{dh}{dy}. \tag{4.6b}$$

The interior pressure and zonal velocity u_I, are determined only up to a function of y. It is important to note that this ambiguity exists solely for the interior, geostrophic, zonal flow. The corresponding flow in the Ekman layer is completely determined. The function $h(y)$ is determined only by consideration of the boundary layers on $x = 0$ and $x = 1$.

5. The meridional boundary layers

On each meridional wall ($x = 0$ and $x = 1$) boundary layers occur which provide correction fields to the interior flow to satisfy the zero boundary conditions on the velocity. In the region below the Ekman layer two distinct regions are found.

First, a hydrostatic layer of thickness $E^{\frac{1}{3}}$, in which the downstream velocity is geostrophic, similar to that found by Munk (1950), is required. This layer is produced by the β effect, i.e. the variation of f.

Interior to this β layer, an upwelling region is found whose dynamics depend on the magnitude of δ. If $\delta \gg E^{\frac{1}{2}}$ this inner region has a thickness $(\delta E)^{\frac{1}{3}}$, is non-hydrostatic, and is in fact dynamically similar to the $\frac{1}{3}$ power layer discussed by Stewartson (1957). If, however, $\delta \ll E^{\frac{1}{2}}$ this inner layer splits into two layers. One has thickness $E^{\frac{1}{2}}$ in which the vertical mass flux balance is achieved. The flow is hydrostatic in this region but the downstream velocity is non-geostrophic. Within this layer an even *thinner non-hydrostatic* region exists of thickness δ which serves only to bring the vertical velocity to rest.

The latter parameter regime, $\delta \ll E^{\frac{1}{2}}$, is probably of greater oceanographic relevance, and since it also presents certain novel dynamical features, the discussion will be limited to that case. The case when $\delta \gg E^{\frac{1}{2}}$ leads to no new qualitative features as far as the interior circulations are concerned.

Consider first the boundary-layer region near the western wall, i.e. near $x = 0$.

Within the outer boundary layer the variables are given as

$$u = E^{\frac{1}{2}}u_I(x,y) + \dots \quad + E^{\frac{1}{2}}\hat{u}(\eta,y) + \dots, \tag{5.1a}$$

$$v = E^{\frac{1}{2}}u_I(x,y) + \dots \quad + E^{\frac{1}{6}}\hat{v}(\eta,y) + \dots, \tag{5.1b}$$

$$w = E^{\frac{1}{2}}w_I(x,y) + \quad E^{\frac{1}{3}}\hat{w}, \tag{5.1c}$$

$$p = E^{\frac{1}{2}}p_I(x,y) + \dots \quad E^{\frac{1}{2}}\hat{p}(\eta,y) + \dots, \tag{5.1d}$$

where $$\eta = xE^{-\frac{1}{3}}.$$

The caret correction functions satisfy:

$$0 = -\hat{p}_\eta + f\hat{v}, \tag{5.2a}$$

$$0 = -\hat{p}_y - f\hat{u} + \tfrac{1}{2}\hat{v}_{\eta\eta}, \tag{5.2b}$$

$$0 = -\hat{p}_z, \tag{5.2c}$$

$$\hat{u}_\eta + \hat{v}_y = 0, \tag{5.2d}$$

and go to zero as $\eta \to \infty$.

Eliminating the pressure yields

$$\hat{v}_{\eta\eta\eta} - 2\beta\hat{v} = 0 \tag{5.3}$$

with solutions,

$$\hat{v} = C(y)\exp\left(-(2\beta)^{\frac{1}{3}}\eta/2\right)\sin\left[(2\beta)^{\frac{1}{3}}\eta\frac{\sqrt{3}}{2}\right], \tag{5.4a}$$

$$\hat{u} = (2\beta)^{-\frac{1}{3}}\frac{dC}{dy}\exp\left(-(2\beta)^{\frac{1}{3}}\eta/2\right)\sin\left[(2\beta)^{\frac{1}{3}}\eta\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right]. \tag{5.4b}$$

This is the ordinary viscous model of the Gulf Stream due to Munk. The function $C(y)$ is determined by matching to the interior.

Within this layer two narrower layers exist as mentioned above. The thicker of these two layers has a thickness $E^{\frac{1}{2}}$ within which the dynamic variables are represented by:

$$u = E^{\frac{1}{2}}u_I + \dots \quad E^{\frac{1}{3}}\hat{u}(\eta, y) + \dots \quad + E^{\frac{1}{2}}\tilde{u}(\xi, y, z) + \dots, \tag{5.5a}$$

$$v = E^{\frac{1}{2}}v_I + \dots \quad E\hat{v}(\eta, y) + \dots \quad + E^{\frac{1}{2}}\tilde{v}(\xi, y, z) + \dots, \tag{5.5b}$$

$$w = E^{\frac{1}{2}}w_I + \dots \quad O(E) + \dots \quad + \tilde{w}(\xi, y, z) + \dots, \tag{5.5c}$$

$$p = E^{\frac{1}{2}}p_I + \dots \quad E^{\frac{1}{2}}\hat{p}(\eta, y) + \dots \quad + E\tilde{p}(\xi, y, z) + \dots, \tag{5.5d}$$

where $\xi = xE^{-\frac{1}{2}}$.

The tilde variables satisfy:

$$0 = -\tilde{p}_\xi + f\tilde{v} + \tfrac{1}{2}\tilde{u}_{\xi\xi}, \tag{5.6a}$$

$$0 = -f\tilde{u} + \tfrac{1}{2}\tilde{v}_{\xi\xi}, \tag{5.6b}$$

$$0 = -\tilde{p}_z, \tag{5.6c}$$

$$\tilde{u}_\xi + \tilde{w}_z = 0. \tag{5.6d}$$

The motion is still hydrostatic but the downstream velocity is no longer geostrophic.

From (5.6d) we may write

$$\tilde{u} = +\tilde{\psi}_z, \quad \tilde{w} = -\tilde{\psi}_\xi.$$

Eliminating all variables in favour of $\tilde{\psi}$ yields

$$\frac{\partial^2}{\partial z^2}\left[\tilde{\psi}_{\xi\xi\xi\xi} + 4f^2\tilde{\psi}\right] = 0,$$

so that

$$\tilde{\psi}_{\xi\xi\xi\xi} + 4f^2\tilde{\psi} = a(\xi) + b(\xi)z. \tag{5.7}$$

To determine the functions $a(\xi)$ and $b(\xi)$ we note first that $\tilde{w} = 0$ on $z = 0$ so that $a(\xi) = 0$. To determine $b(\xi)$ the $E^{\frac{1}{2}}$ by $E^{\frac{1}{2}}$ corner on $x = 0$, $z = 1$ must be analyzed. This is usually an extraordinarily difficult task. Nevertheless, in this problem sufficient information can be obtained from analysis to determine $b(\xi)$. Since the upper surface is a surface of zero stress for the correction functions. equations for the vertical average of the corner corrections may be obtained. Then by satisfying the zero velocity conditions on $x = 0$ in conjunction with the fields given by (3.2) the vertical means are completely determined in the corner. These in turn yield the vertical velocity pumped out of the $E^{\frac{1}{4}}$ side wall layer into the corner. The details are presented in the appendix. The result is that on $z = 1$,

$$E^{\frac{1}{2}}\tilde{\psi}(\xi, y, 1) = e^{-\xi\sqrt{f}}[V_E \sin \xi\sqrt{f} + U_E \cos \xi\sqrt{f}]. \tag{5.8}$$

Substitution of (5.8) into (5.7) yields $b(\xi) = 0$.

Thus
$$E^{\frac{1}{2}}\tilde{\psi}(\xi, y, z) = e^{-\xi\sqrt{f}}[A(z) \sin \xi\sqrt{f} + B(z) \cos \xi\sqrt{f}]. \tag{5.9}$$

It is important to note that this thinner layer, whose mathematical structure is similar to the Ekman layer, and which is absent in the transport theories, can absorb as large a zonal velocity as the outer layer. Thus the matching conditions on the horizontal motions are in reality more involved than indicated by the transport theories. Since the interior horizontal velocities as well as the horizontal velocities in the Munk layer are independent of z we require that $A(z)$ and $B(z)$ be linear functions of z. Then by matching (5.9) with (5.8) at $z = 1$ we obtain for the upwelling layer

$$E^{\frac{1}{2}}\tilde{\psi}(\xi, y, z) = z\, e^{-\xi\sqrt{f}}[V_E(0, y) \sin \xi\sqrt{f} + U_E(0, y) \cos \xi\sqrt{f}]. \tag{5.10}$$

Interior to this layer the thinnest layer exists to bring w to rest. This layer possesses negligible vertical and horizontal mass fluxes, and does not enter into the important matching formulae relating the horizontal velocities. For brevity therefore its analysis will not be presented.

On the eastern wall the structure of the inner layer is unchanged and the correction stream function is of the same magnitude and is represented by

$$E^{\frac{1}{2}}\tilde{\psi}(\mu, y, z) = z\, e^{-\mu\sqrt{f}}[+V_E(1, y) \sin \mu\sqrt{f} + U_E(1, y) \cos \mu\sqrt{f}], \tag{5.11}$$

where $\tilde{u} = \tilde{\psi}_z$, $\tilde{w} = \tilde{\psi}_\mu$ and $\mu = (1-x)E^{-\frac{1}{2}}$.

On the other hand, in the outer layer on the eastern side, the east–west asymmetry due to the β effect changes (5.3) to

$$\hat{v}_{\lambda\lambda\lambda} + 2\beta\hat{v} = 0,$$

where $\lambda = (1-x)E^{-\frac{1}{3}}$. Only a single decaying solution exists, a simple exponential solution, and the no-slip condition on v requires that $\hat{v}$ be $O(E^{\frac{1}{2}})$ rather than $O(E^{\frac{1}{4}})$ as it is on the western wall. This in turn requires that $\hat{u}$ be $O(E^{\frac{5}{6}})$. Consequently on $x = 1$ only the thinner upwelling layer can enter the matching condition on the zonal velocity, i.e.

$$u_I + \tilde{u} = 0 \quad \text{on} \quad x = 1$$

or with (4.6*b*) and (5.11)
$$E^{\frac{1}{2}} u_I(1, y) = -U_E(1, y) \tag{5.12}$$

or
$$\frac{1}{f}\frac{dh}{dy} = \frac{\tau^{(y)}}{2f}(1, y). \tag{5.13}$$

Thus
$$u_I(x, y) = \frac{1}{2\beta f}\frac{\partial}{\partial y} f^2 \int_x^1 dx' \mathbf{k} . \operatorname{curl}\left(\frac{\boldsymbol{\tau}}{f}\right) - \frac{\tau^{(y)}}{2f}(1, y). \tag{5.14}$$

The geostrophic zonal flow is not zero on $x = 1$ unless the wind stress itself is parallel to the x axis at $x = 1$. The total transport, however, *is* zero.

Matching the zonal velocity on the western wall implies that

$$u_I(0,y)+\hat{u}(0,y)+\tilde{u}(0,y) = 0. \tag{5.15}$$

Using (5.14), (5.4*b*), (5.10) and (3.2*d*)

$$\left[\frac{1}{2\beta f}\frac{\partial}{\partial y}f^2\int_0^1 dx'\hat{\mathbf{k}}.\,\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right)-\frac{\tau^{(y)}(1,y)}{2f}\right]+\frac{\tau^{(y)}(0,y)}{2f}+(2\beta)^{-\frac{1}{3}}\frac{\sqrt{3}}{2}\frac{dC}{dy} = 0,$$

or after some manipulation

$$C(y) = -\frac{2}{\sqrt{3}}(2\beta)^{-\frac{2}{3}}\left[\int_0^1 dx'\hat{\mathbf{k}}.\,\mathrm{curl}\,\boldsymbol{\tau}+K\right]. \tag{5.16}$$

The constant K yields a component of the northward boundary-layer transport which is independent of y and can be determined by matching boundary-layer fluxes at the northern or southern corners of the western Munk layer with the horizontal mass flux entering from the boundary layers on $y = 0$ or $y = b$. It is interesting to note that the value of $C(y)$ is the same as given by the transport theories. Thus the strength of the western boundary current, which we identify with the Gulf Stream, is independent of the partitioning of the interior flow between the Ekman layer transport and the geostrophic transport. This has interesting consequences which will be discussed later.

6. The latitudinal boundary layers

On both $y = 0$ and $y = b$ we again find a multiple layer structure. There is again an upwelling layer (which we now see girdles the entire basin) which once again serves to accept the Ekman flux normal to the boundary allowing it to descend to complete the mass flux circuit. The outer hydrostatic β layer is now even thicker on $y = 0$ and $y = b$; its thickness is $E^{\frac{1}{4}}$. The northward velocity is small in this layer, $O(E^{\frac{1}{2}})$, and the consequent reduction in the advection of planetary vorticity explains the more diffuse character of the layer.

In the outer layer the various fields have the following representations. On the northern boundary, for example,

$$u = E^{\frac{1}{2}}u_I(x,y)+\ldots . \quad E^{\frac{1}{4}}\hat{u}(x,\eta)+\ldots, \tag{6.1a}$$

$$v = E^{\frac{1}{2}}v_I(x,y)+\ldots . \quad E^{\frac{1}{2}}\hat{v}(x,\eta)+\ldots, \tag{6.1b}$$

$$w = E^{\frac{1}{2}}w_I(x,y)+\ldots . \quad E^{\frac{1}{2}}\hat{w}(x,\eta,z)+\ldots, \tag{6.1c}$$

$$p = E^{\frac{1}{2}}p_I(x,y)+\ldots . \quad E^{\frac{1}{2}}\hat{p}(x,\eta)+\ldots, \tag{6.1d}$$

where the boundary-layer correction variables, denoted by a caret, go to zero as $\eta = (b-y)E^{-\frac{1}{4}}$ becomes large.

The correction fields satisfy

$$0 = +\hat{p}_\eta - f\hat{u}, \tag{6.2a}$$

$$0 = -\hat{p}_x + f\hat{v} + (\tfrac{1}{2}E^{\frac{1}{4}})\hat{u}_{\eta\eta}, \tag{6.2b}$$

$$0 = -\hat{p}_z, \tag{6.2c}$$

$$\hat{u}_x - \hat{v}_\eta = -\hat{w}_z E^{+\frac{1}{4}}. \tag{6.2d}$$

On $z = 0$ the Ekman layer provides a suction given by the relation

$$\hat{w}(x, \eta, 0) = \frac{1}{2\sqrt{f}} \frac{\partial \hat{u}}{\partial \eta}. \tag{6.2e}$$

If all variables are eliminated in favour of the pressure we obtain

$$\hat{p}_{\eta\eta\eta\eta} - \sqrt{f}\,\hat{p}_{\eta\eta} - 2\beta \hat{p}_x = 0. \tag{6.3}$$

Note that the vertical flux in this layer is $O(E^{\frac{2}{3}})$ which is too small to figure in the vertical mass flux balance. Again it is the upwelling layer which must accept the northward Ekman flux on $y = b$. The analysis of the inner layer will proceed as before. For the sake of brevity it will not be repeated, rather the results will be quoted as needed.

Boundary conditions for (6.3) are obtained as follows.

There is no layer on $x = 1$ which will provide an $O(E^{\frac{1}{2}})$ horizontal mass flux. Therefore the total mass flux impinging on the northern wall must turn and proceed to the western boundary layer for its trip south to complete the horizontal mass flux circuit. This implies that for each x on $y = b$,

$$E^{-\frac{1}{2}}(V_I + V_E) = -\int_0^\infty \hat{u}\, d\eta = \frac{1}{2\beta}\int_x^1 \hat{\mathbf{k}}.\operatorname{curl} \boldsymbol{\tau}\, dx' = \frac{\hat{p}(0, x)}{f}. \tag{6.4}$$

The no slip condition on u requires that

$$\partial \hat{p}/\partial \eta = 0 \quad \text{on} \quad y = b. \tag{6.5}$$

With the condition $$\hat{p}(1, \eta) = 0,$$

sufficient conditions are available for the solution of (6.3).

The solution of (6.3) is quite complicated and is not presented here. The boundary layer has the character of a diffusion-like equation with the origin of the time like variable, x, at $x = 1$. The layer grows in thickness as $x = 0$ is approached from the east.

From (6.4) we see that on $y = b$,

$$\hat{\mathbf{v}} = \frac{1}{f}\frac{\partial \hat{p}(x, 0)}{\partial x} = \frac{1}{2\beta}\hat{\mathbf{k}}.\operatorname{curl} \boldsymbol{\tau}. \tag{6.6}$$

The matching condition on v at $y = b$ requires that the northward correction velocity in the upwelling layer, $\tilde{v}$, must satisfy

$$\begin{aligned} \tilde{v} &= -v_I(x, b) - \hat{v}(x, 0) \\ &= -\frac{f}{2\beta}\hat{\mathbf{k}}.\operatorname{curl}\frac{\boldsymbol{\tau}}{f} + \frac{1}{2\beta}\hat{\mathbf{k}}.\operatorname{curl}\boldsymbol{\tau}, \quad \text{at} \quad y = b, \end{aligned} \tag{6.7}$$

or that $$\tilde{v}(x, 0) = -\frac{\tau^{(x)}(x, b)}{2f} = V_E(x, b)\, E^{-\frac{1}{2}}, \tag{6.8}$$

which matches the result yielded by detailed analysis of the upwelling layer.

Thus, the $E^{\frac{1}{4}}$ layer closes the horizontal mass flux while once again the upwelling layer transfers the flux in the upper Ekman layer downward and returns it to the interior.

The constant K, in the solution in the western layer, can now be determined.

The total northward mass flux in the western boundary current on $y = b$, V_B is

$$V_B = \frac{\sqrt{3}}{2}(2\beta)^{-\frac{1}{3}}C(b) = \frac{-1}{2\beta}\left[\int_0^1 dx'(\hat{\mathbf{k}}.\operatorname{curl}\boldsymbol{\tau})_{y=b}+K\right],$$

while the eastward mass flux in the northern boundary layer at $x = 0$ is

$$U_B = -\frac{1}{2\beta}\int_0^1 dx'(\hat{\mathbf{k}}.\operatorname{curl}\boldsymbol{\tau})_{y=b}.$$

Matching fluxes in the north-west corner requires that

$$K = 0, \tag{6.9}$$

which completes the problem.

7. Examples

Now that the matching is complete, it is of interest to consider some special examples. The important point to be kept in mind is that the vertical flux balance achieved in the upwelling layers imposes compatibility conditions on the geostrophic flow which is manifested by certain interior circulations in addition to those driven by the Ekman layer flux divergence (suction).

Example 1. $\boldsymbol{\tau} = T(y)\hat{\mathbf{j}}$

In this example, with a wind stress directed solely northward and independent of x, both $\hat{\mathbf{k}}.\operatorname{curl}\boldsymbol{\tau}$ and $\hat{\mathbf{k}}.\operatorname{curl}(\boldsymbol{\tau}/f)$ are zero. Thus the transport theories would reveal only that no net transport is occurring and would yield no other information. Nevertheless, a very interesting circulation exists. From (3.2d) we see that there is a zonal flux directed to the east in the Ekman layer,

$$\mathbf{U}_E = \hat{\mathbf{i}}E^{\frac{1}{2}}\frac{T(y)}{2f}. \tag{7.1a}$$

In the interior there is no northward flux, nor is there any Ekman layer suction so that $w_I = 0$ also. Nevertheless, there is a zonal flow given by (5.13b),

$$u_I = -\frac{1}{2f}T(y), \tag{7.1b}$$

which just compensates for the Ekman layer flux. There are no β layers and hence no western intensification due to the zonal flow given by (6.1b). The entire flux circuit is confined to the (x,z)-plane. The fluid moves eastward in the upper Ekman layer, down the eastern upwelling (here a downwelling) layer, westward in the interior and upward in the western upwelling layer. The geostrophic circulation is the same order as those predicted by the transport theory, but a transport theory would not reveal the presence of this circulation. Note again that the circulation has no east–west asymmetry.

Example 2. $\boldsymbol{\tau} = -A(x)f\hat{\mathbf{i}}$

In this case the wind stress is strictly zonal. The northward geostrophic transport v_I is identically zero and the entire northward flux,

$$V = E^{\frac{1}{2}}\frac{\hat{\mathbf{k}}}{2\beta}.\operatorname{curl}\boldsymbol{\tau} = E^{\frac{1}{2}}A(x) = V_E,$$

takes place in the Ekman layer. Upon reaching $y = b$, the fluid descends in the upwelling layer and flows westward in the $E^{\frac{1}{4}}$ layer and then southward in the western β layer. It then flows eastward in the southern $E^{\frac{1}{4}}$ layer while rising in the upwelling layer to enter the southern edge of the Ekman layer to complete the circuit.

Thus although there is no flow in the interior below the Ekman layer, there is a transport in the northern, western and southern boundary currents extending to the bottom. The vertical average of this circulation is identical, of course, to that predicted by the transport theories.

Example 3. $\boldsymbol{\tau} = -B(x)\hat{\mathbf{i}}$

The circulation in this case is similar to the first example. There is no vertically averaged transport. The flow goes northward in the Ekman layer, descends in the northern upwelling layer, proceeds southward in the interior and ascends in the southern upwelling layer to complete the circuit in a roll type circulation with again, no western intensification.†

8. Conclusion

In general the circulations predicted, even in this very simple model, are much more complex than indicated by theories dealing only with the horizontal mass transport. In addition to the β boundary layers which are required to close the horizontal transports and which introduce the pronounced east-west asymmetry in the ocean circulation, narrow upwelling regions girdling the entire basin are required to achieve a balance of the vertical mass flux. This balance of the vertical mass flux forces interior geostrophic circulations in response to the Ekman layer flux. It is somehow satisfying to see the reappearance of the stress itself as an important factor in determining the large scale interior circulations in addition to the curl of the stress.

If β is small the circulations produced by the stress itself will be smaller by $O(\beta)$ than the circulations produced by the curl of the stress. Nevertheless, even in such cases it may be necessary to consider this smaller circulation to fully understand the closure of the fluid circuit.

For example H. P. Greenspan and I considered a related problem in which fluid in a rotating cylinder with a slanted bottom is driven by a differential motion of the upper surface (Pedlosky & Greenspan 1967). In the example studied, the bottom slope (analogous to β) was small. The fluid in the Ekman layer was flung out radially with complete axial symmetry while the largest downward velocities were restricted to the 'western' side of the cylinder. It was something of a puzzle to explain how the circulation circuit of a fluid element which impinged on the 'eastern' boundary in the Ekman layer was completed. Although the model differs in some respects from the one discussed in this paper, it is clear that a smaller circulation of the type found in example 1 of the preceding section must be added to complete the fluid circuit.‡

† Due to the β effect the Ekman layer leaks fluid to the interior, (equation (3.2*e*)), so that only a portion of the flow descends in the $E^{\frac{1}{2}}$ layer on $y = b$.

‡ This circulation has recently been calculated by Derek Moore & Phillip Saffman (private communication).

Finally, these additional circulations in the interior, forced by the Ekman layer flux, have important implications for the case of a stratified ocean, where this additional transport of heterogeneous fluid will provide buoyancy effects and contribute to the oceanic heat balance. This problem is being studied further.

My interest in the problem was stimulated greatly by my discussions with Victor Barcilon on the problem of the circulation of a stratified ocean.

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Appendix

The analysis of the $E^{\frac{1}{2}} \times E^{\frac{1}{2}}$ corner proceeds as follows.

In the corner at $z = 1$, $x = 0$, for example, let

$$\zeta = (1-z)E^{-\frac{1}{2}}, \quad \xi = xE^{-\frac{1}{2}}.$$

Then in the corner the fields have the following representations:

$$u = u_E(x,y,\zeta)+\dots \qquad u_c(\xi,y,\zeta)+\dots \quad O(E^{\frac{1}{2}}), \tag{A 1a}$$

$$v = v_E(x,y,\zeta)+\dots \qquad v_c(\xi,y,\zeta)+\dots \quad O(E^{\frac{1}{2}}), \tag{A 1b}$$

$$w = \qquad w_c(\xi,y,\zeta)+\dots \quad O(E^{\frac{1}{2}}), \tag{A 1c}$$

$$p = E^{\frac{1}{2}}p_E(x,y,\zeta)+\dots \quad \delta^2 E^{\frac{1}{2}} p_c(\xi,y,\zeta)+\dots, \tag{A 1d}$$

where the c subscripted variables are the corner correction functions and satisfy

$$0 = +fv_c+\tfrac{1}{2}(u_{c\xi\xi}+u_{c\zeta\zeta}), \tag{A 2a}$$

$$0 = -fu_c+\tfrac{1}{2}(v_{c\xi\xi}+v_{c\zeta\zeta}), \tag{A 2b}$$

$$u_{c\xi} = w_{c\zeta}, \tag{A 2c}$$

subject to the boundary conditions

$$\tilde{w}+w_c = u_{c\zeta} = v_{c\zeta} = 0 \quad \text{on} \quad \zeta = 0, \tag{A 3a}$$

$$u_c = -u_E \quad \text{on} \quad \xi = 0, \tag{A 3b}$$

$$v_c = -v_E \quad \text{on} \quad \xi = 0, \tag{A 3c}$$

while the corrections go to zero as ζ and ξ become large. By vertically integrating the system (A 2) we find with the use of (A $3a$) that

$$0 = fV_c+\tfrac{1}{2}U_{c\xi\xi}, \tag{A 4a}$$

$$0 = -fU_c+\tfrac{1}{2}V_{c\xi\xi}, \tag{A 4b}$$

$$U_{c\xi} = -w_c(\xi,0), \tag{A 4c}$$

where
$$U_c = E^{\frac{1}{2}}\int_0^\infty u_c d\zeta, \quad V_c = E^{\frac{1}{2}}\int_0^\infty v_c d\zeta.$$

The solution of (A 4) subject to (A $3b$) and (A $3c$), (or their vertical integrals) is

$$U_c = -e^{-\xi\sqrt{f}}[V_E \sin\sqrt{f}\,\xi + U_E\cos\sqrt{f}\,\xi], \tag{A 5a}$$

$$V_c = -e^{-\xi\sqrt{f}}[V_E \cos\sqrt{f}\,\xi - U_E\sin\sqrt{f}\,\xi], \tag{A 5b}$$

and with (A 3*a*)
$$-w_c(\xi, 0) = \tilde{w}(\xi, 1) = \frac{\partial}{\partial \xi} U_c,$$

or since
$$\tilde{w}(\xi, 1) = -\tilde{\psi}_\xi(\xi, 1),$$

we find
$$\tilde{\psi}(\xi, 1) = e^{-\sqrt{f}\xi}[V_E \sin \sqrt{f}\, \xi + U_E \cos \sqrt{f}\, \xi] \tag{A 6}$$

which is identical to (5.10).

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